Topology-B.Math II year
Semestral Examination

03.06.2006

Answer any *six* questions. State clearly any result that you use.

1. Let (X, τ) be a compact Hausdorff space. Let τ_1 be a topology on X which is strictly finer than τ and τ_2 a topology on X which is strictly coarser than τ. Show that (X, τ_1) is not compact and (X, τ_2) is not Hausdorff. [10]

2. Show that $\mathbb{R}^\omega$ (the countable product of $\mathbb{R}$ with itself) with product topology cannot be written as a countable union of compact subspaces. Conclude that there does not exist a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^\omega$ which is surjective. [10]

3. Let (X, d) be a metric space. Suppose that for some $\varepsilon > 0$ every ε-ball has compact closure. Show that (X, d) is complete. Show by an example that the conclusion is false under the assumption : for each $x \in X$ there is an $\varepsilon_x > 0$ such that $B_d(x, \varepsilon_x)$ has compact closure. [10]

4. Define the term paracompact. Let X be a regular space. If X is a countable union of compact subspaces, show that X is paracompact. Conclude that the subspace $\mathbb{R}^\infty = \{(x_i) | x_i = 0 \text{for all but finitely many } i\}$ of $\mathbb{R}^\omega$ is paracompact. [10]

5. Let X be a non-compact normal space. Let $\beta(X)$ denote its Stone-Cech compactification. Let $y \in \beta(X) - X$. Show that there does not exist a sequence in X converging to y. Conclude that $\beta(X)$ is not metrizable. [10]

6. Let X be a topological space and (Y, d) be a metric space. Show that $C(X, Y)$ and $B(X, Y)$, the set of all continuous functions and bounded functions respectively from X to Y are closed in Y^X in the uniform metric. [10]

7. State Ascoli's theorem. Define all the terms in the statement of the theorem. For $n \geq 1$, let $f_n : [0, 1] \longrightarrow \mathbb{R}$ be the function $f_n(x) = x^n$. Show that the closure of the collection $\{f_n\}_{n \geq 1}$ is not compact as a subspace of $C([0.1], \mathbb{R})$ with the uniform topology. [10]

8. When do you say a space is contractible. Show that a contractible space is path connected. Show by an exapmple that a contractible space need not be locally path connected. [10]

9. When do you say a subspace A of a space X is a deformation retract of X. Show that S^1 is a deformation retract of $\mathbb{R}^2 - 0$. Conclude that the inclusion map $i : S^1 \longrightarrow \mathbb{R}^2 - 0$ is not null homotopic. [10]